An Alternative Derivation of the Linear Inverted Pendulum Model

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The Linear Inverted Pendulum model is a well-known and popular approach for biped gait planing as it provides the relation between the Zero Moment Point (ZMP) based on a single center of mass (CoM). It is linear and can therefore be applied to various approaches to generate a motion of the CoM given a desired ZMP trajectory. However, for beginners in research or in education the derivation as presented in [1] can be confusing and presents many details that are not required.

Here we intend to present two derivations. The first is a intuitive approach to the ZMP by explaining the concept using a beam balance. In the following we derive the ZMP of the Linear Inverted Pendulum model as presented in [1].

Zero Moment Point of a Multi-Body System

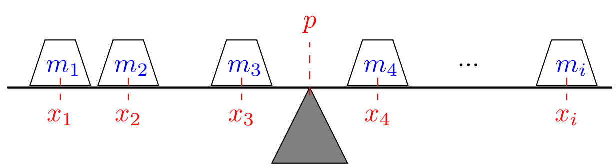

Figure 1: Beam balance with multiple masses.

The ZMP as defined by Vukobratović [2] is a result of torques and forces due to accelerations which includes gravity. In a simplified manner we now assume that all accelerations are caused by gravity to explain the concept of the ZMP utilizing a beam balance as depicted in Fig. 1. We search for the coordinate[1] of the Zero Moment Point p where the sum of all torques is 0. If our base supports the beam at this point, the beam is stable. For all $k \in \{1, \dots, i\}$ the force is $f_k = m_k \cdot g$, where g is the gravity. Thus, the base must support the beam with a total force of $F = \sum f_k$ at point p leading to the equation of all torques acting in this system:

$$F \cdot p = \sum (f_k \cdot x_k) \Leftrightarrow p = \frac{\sum (f_k \cdot x_k)}{F}$$

Zero Moment Point of the Linear Inverted Pendulum Model

We now consider the ZMP to be at $p = 0$. Comparably to the derivation above the sum of all torques in this system must be zero at $p = 0$: It follows that $\tau_1 = \tau_2$. Due to gravity we have $\tau_1 = m \cdot g \cdot x$, where m is the mass, x the one-dimensional position of the mass and g gravity. To realize τ_2 the system must be accelerated and it follows $\tau_2 = m \cdot \ddot{x} \cdot z_h$ where z_h must be constant to neglect dynamics of changing height. Rearranging leads to:

$$\begin{aligned} \tau_1 &= \tau_2 & \Leftrightarrow \\ m \cdot g \cdot x &= m \cdot \ddot{x} \cdot z_h & \Leftrightarrow \\ 0 &= x - \frac{z_h}{g}\ddot{x} \end{aligned}$$

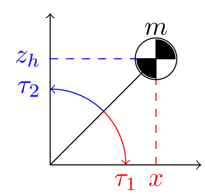

Figure 2: Linear Inverted Pendulum Model

References

[1] Kajita, S., Kanehiro, F., Kaneko, K., Fujiwara, K., Yokoi, K., Hirukawa, H.: A realtime pattern generator for biped walking. In: ICRA. pp. 31–37. IEEE (2002)

[2] Vukobratović, M., Borovac, B.: Zero-moment point – Thirty five years of its life. International Journal of Humanoid Robotics **1**(1) (2004) 157–173

[1] Please note that the ZMP must be on the floor and is thus one-dimensional in this planar example.